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Transitional description of mass spectra and radiative decay widths for $q\bar{q}$ mesons in the U(4) model

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Abstract. Mass spectra and radiative decay widths for $q\bar{q}$ mesons taken from well-established results of the Particle Data Group are fit by the $O(4) \longleftrightarrow U(3)$ transitional theory of the $U(4)$ vibron model. The results are compared with those of the $O(4)$ limit of the theory and the quark potential model. The $O(4) \longleftrightarrow U(3)$ transitional theory seems to give a better description of the heavy $q\bar{q}$ mesons, while the $O(4)$ limit gives an accurate description of the light and strange $q\bar{q}$ mesons.

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1 Introduction

The $U(4)$ vibron model has been used successfully to characterize the rotational and vibrational motion of complex systems [1–4]. The model was first used to describe the rotation-vibration spectra of diatomic molecules [2]. Then, Iachello *et al.* in [5] adopted the $U(4)$ structure to describe the quantized geometric excitations of the string-like $q\bar{q}$ mesons. As was shown clearly in [5], the $U(4)$ vibron model is a simple algebraic model to describe the $q\bar{q}$ configuration performing rotations and vibrations leading to the spatial excitations. The spectrum-generating algebra for a $q\bar{q}$ meson system is given by

$$\mathcal{L} = U(4) \otimes SU_s(2) \otimes SU_f(6) \otimes SU_c(3), \qquad (1)$$

where the subscripts s, f, and c denote spin, flavor, and color, respectively. In this theory, the $U(4)$ algebra can be decomposed into two chains that contain the angular momentum $SO(3)$ algebra as a subalgebra:

$$\begin{array}{ll} U(4) \supset U(3) \supset SO(3) \supset O(2), & \text{(I)} \\ U(4) \supset O(4) \supset SO(3) \supset O(2). & \text{(II)} \end{array} \qquad (2)$$

Generally, a model Hamiltonian should be diagonalized in the full $U(4)$ symmetric bosonic space. As a simplification, Iachello *et al.* in [5] argued that Chain (II) is appropriate for a description of $q\bar{q}$ mesons because the observation of rotational trajectories for mesons can be accommodated within a single representation of $O(4)$. Also, as is generally known, Chain (I) is appropriate for problems involving harmonic-oscillator potentials, while Chain (II) can be used to simulate Coulomb-like or linear potentials. The combination of the two is a suggested QCD form. According to the results shown in [5], it is found that spatial rotation-vibration excitations in the $U(4)$ vibron model should be more important than intrinsic spin excitations in low-lying meson spectra. It is possible that the space spanned by the $O(4)$ limiting case is insufficient in the description of low-lying spatial excitations. Hence, whereas the $O(4)$ limit of the theory, which was adopted in [5], simplifies the theory greatly and describes certain experimental data rather well, the use of the full $O(4) \longleftrightarrow U(3)$ transitional theory is thus motivated, and is a natural extension of the special $O(4)$ limit of the theory.

Recently, the $O(4) \longleftrightarrow U(3)$ transitional description of diatomic molecules in the $U(4)$ vibron model has been shown to yield a better description of the data than the simpler $O(4)$ limit of the theory [6]. The analysis indicates that there are notable deviations from the $O(4)$ limit for certain diatomic molecules, a result that encouraged us to explore whether or not the transitional theory can be used to provide a better description of $q\bar{q}$ mesons. The paper is organized as follows. In sect. 2, a brief outline of the transitional theory of the $U(4)$ model for describing spatial string excitations of $q\bar{q}$ mesons is given. In sect. 3, 158 meson masses taken from the Particle Data Group

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(PDG) and [7] for some excited states [8] are fitted and compared to results obtained from the $O(4)$ limit of the theory and quark potential model. Radiative decay widths of $q\bar{q}$ mesons are reported in sect. 4. Finally, a brief synopsis and some conclusions are offered in sect. 5.

2 Mass formula in the transitional theory of the U(4) model

In the $U(4)$ vibron model, elementary spatial excitations are dipole p-bosons with spin and parity $l^P = 1^-$ and scalar s-bosons with $l^P = 0^+$. Since the total number of bosons and angular momentum are conserved quantities, the leading dynamical symmetry group is $U(4)$. Two possible dynamical symmetry limits, $U(3)$ and $O(4)$, can be realized when the Hamiltonian of a system is exactly diagonal in the basis of one of the algebraic chains given in (2). Similar to [5], if only one- and two-body interactions are considered, according to the full spectrum-generating algebra (1) the mass-squared operator M^2 of $q\bar{q}$ meson in the $U(3) \longleftrightarrow O(4)$ transitional theory may be constructed in terms of the Casimir operators of all subalgebras shown in (2), which can be written as

$$\begin{aligned} M^2 = {} & \frac{A}{N+1} S_0^+ S_0^- + \frac{2M_0^2}{N+2} S_0^0 \\ & + B\left\{\left[C^{(2)}_{SO_L(3)} + \frac{1}{4}\right]^{\frac{1}{2}} - \frac{1}{2}\right\} \\ & + C\left\{\left[C^{(2)}_{SU_S(2)} + \frac{1}{4}\right]^{\frac{1}{2}} - \frac{1}{2}\right\} \\ & + D\left\{\left[C^{(2)}_{SU_J(2)} + \frac{1}{4}\right]^{\frac{1}{2}} - \frac{1}{2}\right\}, \end{aligned} \tag{3}$$

where $C^{(2)}_{SO_L(3)}$, $C^{(2)}_{SU_S(2)}$, and $C^{(2)}_{SU_J(2)}$ are Casimir operators of $SO(3)$ for the orbital angular momentum and $SU(2)$ for the spin and total angular momentum, respectively,

$$S_0^{\pm} = cS^{\pm}(s) + S^{\pm}(p), \qquad S_0^0 = c^2 S^0(s) + S^0(p) \tag{4}$$

with $S^+(s) = \frac{1}{2}s^{\dagger 2}$, $S^-(s) = \frac{1}{2}s^2$, $S^0(s) = \frac{1}{2}(s^\dagger s + \frac{1}{2})$, $S^+(p) = \frac{1}{2}p^\dagger \cdot p^\dagger$, $S^-(p) = \frac{1}{2}\tilde{p} \cdot \tilde{p}$, $S^0(p) = \frac{1}{2}(p^\dagger \cdot \tilde{p} + \frac{3}{2})$, and A, B, C, D, M_0^2, and c are parameters of the theory. It is obvious that the system is in the $U(3) \longleftrightarrow O(4)$ transitional region as c varies continuously in the closed interval [0, 1], which includes the $U(3)$ and $O(4)$ limits as special cases when $c = 0$ and $c = 1$, respectively. As is noted in [5], the mass-squared operator adopted in (3) is more appropriate for relativistic situations, of which the linear dependence on the quantum number L instead of $L(L+1)$ is a crucial property of soft QCD strings. It is also noted in [5] that the quantum number of the total number of bosons should be taken in the $N \to \infty$ limit. Also, as exploited in [5], in applications it is sufficient to take N large enough to include all known and unknown states up to a maximum value of the quantum number of the angular momentum L and other quantum numbers associated with it. In the present study, we take this to be the same as that used in [5] with $N = 100$. It should be emphasized that there is only one more parameter c in the new transitional theory in comparison to the $O(4)$ limit case studied in [5]. The Hamiltonian is identical to that of the $O(4)$ limit situation proposed in [5] when c is taken to be 1. Though there is only one more parameter c is adjustable in the transitional theory, the Hamiltonian (3) will no longer be diagonal in the $O(4)$ limit subspace, but should be diagonalized in the full $U(4)$ space.

Let $a^\dagger_{\alpha,m,i}$ ($\bar{a}^\dagger_{\alpha,m,i}$) be creation operators for quarks (antiquarks) with color component α, spin m, and flavor i. The meson state vectors in the transitional theory can be expressed as

$$\begin{aligned} & \left|q_i, \bar{q}_j, N, \xi, L, S, J, M_J\right\rangle = \\ & \sum_{\mu m m' M_S M_L} C^\xi_\mu \left\langle \frac{1}{2}, m, \frac{1}{2}, m' \middle| S, M_S \right\rangle \langle L, M_L, S, M_S | JM \rangle \\ & \times \left[S^+(p)\right]^{(k-\mu)} \left[S^+(s)\right]^{\mu} \left(a^\dagger_{\alpha,m,i} \bar{a}^\dagger_{\alpha',m',j}\right)^{[0]} |\mathrm{lw}\rangle, \end{aligned} \tag{5}$$

where $|\mathrm{lw}\rangle$ stands for the boson pair and quark (antiquark) vacuum state $|\mathrm{lw}\rangle = |N_0 = L + \nu_s, \nu_s, LM_L\rangle$ with s-boson seniority number $\nu_s = 0$ or 1, k is the total number of boson pairs, the total number of bosons is $N = N_0 + 2k = L + \nu_s + 2k$, C^ξ_μ is the expansion coefficient, the additional quantum number ξ is introduced to distinguish different eigenstates with the same number of boson pairs and other quantum numbers, and $(a^\dagger_{\alpha,m,i} \bar{a}^\dagger_{\alpha',m',j})^{[0]}$ stands for a color singlet coupling of a quark and antiquark pair.

As for the $O(4)$ limit discussed in [5], the mass-squared operator of $q\bar{q}$ mesons can be diagonalized under (5) for given L, S, and J when the flavor-dependent parameters $A = A_{ij}$, $B = B_{ij}$, $C = C_{ij}$, $D = D_{ij}$, and $M_0^2 = (M_0^2)_{ij}$ are determined. We adopt the same relations as those used in [5] with

$$\begin{aligned} & A_{ij} = a + a' M_{ij}, \qquad B_{ij} = b + b' M_{ij}, \\ & C_{ij} = \bar{c} + \bar{c}' M_{ij}, \qquad D_{ij} = d + d' M_{ij}, \\ & \left(M_0^2\right)_{ij} = e M_{ij} + \left(M_{ij}\right)^2, \end{aligned} \tag{6}$$

where $M_{ij} = M_i + M_j$, M_i and M_j are the constituent masses of quarks with flavor component i and j, respectively, which are taken from [5,9–11], a, a', b, b', $\bar{c}$, $\bar{c}'$, and e are parameters.

The mass formula obtained by diagonalizing (3) in the basis (5) with flavor parameters introduced in (6) describes the experimental data fairly well except for the pseudo-scalar nonet mesons. In order to improve the results, two correction terms should be introduced and diagonalized in the flavor space. One is

see eq. (7) *on the next page*

where f is a parameter, and δ_8 means that this term is restricted [5] to be the ground-state octet of $SU_f(3)$, which

$$\left\langle q_i\bar{q}_j;N,\xi,L,S,J\middle|M'^2\middle|q_{i'}\bar{q}_{j'};N,\xi',L',S',J'\right\rangle = -f\delta_{\xi 0}\delta_{L0}\delta_{S0}\delta_{J0}\delta_{\xi' 0}\delta_{L'0}\delta_{S'0}\delta_{J'0}\begin{cases}\langle q_i,\bar{q}_j|\delta_8|q_{i'},\bar{q}_{j'}\rangle, & \text{if } i,j,i',j'=u,d,s,\\ 0, & \text{otherwise,}\end{cases} \tag{7}$$

$$\left\langle q_i\bar{q}_j;N,\xi,L,S,J\middle|M''^2\middle|q_{i'}\bar{q}_{j'};N,\xi',L',S',J'\right\rangle = h\delta_{\xi 0}\delta_{L0}\delta_{S0}\delta_{J0}\delta_{\xi' 0}\delta_{L'0}\delta_{S'0}\delta_{J'0}\begin{cases}\delta_{ij}\delta_{i'j'}, & \text{if } i,j,i',j'=u,d,s,\\ 0, & \text{otherwise,}\end{cases} \tag{8}$$

$$R_i = \begin{pmatrix}\cos\theta_i\cos\psi_i-\sin\theta_i\sin\psi_i\cos\phi_i & -\cos\theta_i\sin\psi_i-\sin\theta_i\cos\psi_i\cos\phi_i & \sin\theta_i\sin\phi_i\\ \sin\theta_i\cos\psi_i+\cos\theta_i\sin\psi_i\cos\phi_i & -\sin\theta_i\sin\psi_i+\cos\theta_i\cos\psi_i\cos\phi_i & -\cos\theta_i\sin\phi_i\\ \sin\psi_i\sin\phi_i & \cos\psi_i\sin\phi_i & \cos\phi_i\end{pmatrix}. \tag{12}$$

Table 1. Meson families [5].

Name	Notation
Light unflavored ($I=1$)	π family
Light unflavored ($I=0$)	η family
Strange	K family
Charmed	D family
Charmed strange	D_s family
Bottom	B family
Bottom strange	B_s family
Bottom charmed	B_c family
J/ψ	ψ family
$b\bar{b}$	Υ family

relates to the fact that the π, K and octet combination of η and η' have unusually low masses [5,12]. Another is

see eq. (8) *above*

where h is a parameter, which arises from the fact the quark-antiquark pair with the same flavor quantum numbers can virtually annihilate into gluons and reappear as another $q\bar{q}$ pair [5,12]. Once the above two correction terms are added, the fitting results to the low-lying mass spectra are greatly improved [5]. Hence, the matrix elements of final squared-mass operator in the spatial plus flavor space are

$$\left\langle q_i,\bar{q}_j,N,\xi\middle|M^2\middle|q_{i'},\bar{q}_{j'},N,\xi'\right\rangle = \delta_{ii'}\delta_{jj'}\delta_{\xi\xi'}\left\langle M^2\right\rangle_{\xi\xi'}+\left\langle M'^2\right\rangle_{ij\xi,i'j'\xi'}+\left\langle M''^2\right\rangle_{ij\xi,i'j'\xi'}. \tag{9}$$

3 Mass spectra of $q\bar{q}$ mesons in the transitional theory

Mesons are classified into eight families shown in table 1, which are the same as those given in [5]. There are sixteen parameters a, a', b, b', $\bar{c}$, $\bar{c}'$, d, d', c, e, f, h, $M_u=M_d$, M_s, M_c, and M_b, and two sets of mixing angles $\{\theta_P,\psi_P,\phi_P\}$, and $\{\theta_V,\psi_V,\phi_V\}$ in the mass formula for $q\bar{q}$ mesons calculated according to (9), in which the constituent masses of quarks are taken from [5,9–11] shown in table 2.

It is well known that flavor mixing in mass eigenstates should be considered, which is important to improve both the mass spectra and radiative decay widths. Especially, radiative decays are sensitive to flavor mixings due to the Okubo-Zweig-Iizuka (OZI) suppression, which provide a way to analyze η, η', η_c and η_b and the ω, ϕ, J/ψ, and Υ mixings. Experimentally, for the heavy quark states, only the decays from charmonium are known. Therefore, we only consider explicitly the mixing of η, η', η_c, and ω, ϕ, J/ψ with

$$\begin{pmatrix}\omega\\ \phi\\ J/\psi\end{pmatrix} = R_V\begin{pmatrix}\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\\ s\bar{s}\\ c\bar{c}\end{pmatrix}, \tag{10}$$

$$\begin{pmatrix}\eta\\ \eta'\\ \eta_c\end{pmatrix} = R_P\begin{pmatrix}\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\\ s\bar{s}\\ c\bar{c}\end{pmatrix}, \tag{11}$$

where R_P and R_V are the mixing matrices for pseudoscalar and vector mesons with respect to the constituent quark basis, respectively [13]. We take the mixing matrices R_i ($i=P,V$) to be real and orthogonal with

see eq. (12) *above*

Thus, the final meson states (10) and (11) can be expressed as

$$\begin{aligned}|\omega\rangle &= y^V_{11}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^V_{12}|s\bar{s}\rangle + y^V_{13}|c\bar{c}\rangle,\\ |\phi\rangle &= y^V_{21}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^V_{22}|s\bar{s}\rangle + y^V_{23}|c\bar{c}\rangle,\\ |J/\psi\rangle &= y^V_{31}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^V_{32}|s\bar{s}\rangle + y^V_{33}|c\bar{c}\rangle,\end{aligned} \tag{13}$$

$$\begin{aligned}|\eta\rangle &= y^P_{11}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^P_{12}|s\bar{s}\rangle + y^P_{13}|c\bar{c}\rangle,\\ |\eta'\rangle &= y^P_{21}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^P_{22}|s\bar{s}\rangle + y^P_{23}|c\bar{c}\rangle,\\ |\eta_c\rangle &= y^P_{31}\frac{1}{\sqrt{2}}|u\bar{u}+d\bar{d}\rangle + y^P_{32}|s\bar{s}\rangle + y^P_{33}|c\bar{c}\rangle,\end{aligned} \tag{14}$$

where y^i_{mn} ($m,n=1,2,3$; $i=P,V$) are elements of the orthogonal matrix R_i given by (12).

Table 2. Parameters of the mass-squared formula determined in the fitting.

Parameter	M_u	M_s	M_c	M_b	a	a'	b
Value	0.250 GeV	0.400 GeV	1.510 GeV	4.710 GeV	0.995 GeV^2	0.727 GeV	0.828 GeV^2
Parameter	b'	$\bar{c}$	$\bar{c}'$	d	d'	f	h
Value	0.449 GeV	0.152 GeV^2	0.025 GeV	−0.027 GeV^2	0.135 GeV	0.366 GeV^2	0.097 GeV^2
Parameter	e	θ_p	ψ_p	ϕ_p	θ_v	ψ_v	ϕ_v
Value	0.270 GeV	37.2°	−4.2°	−1.6°	3.4°	0.9°	1.9°

Table 3. The parameter c fitted in the transitional theory for different meson families.

Family	π, η and K familes	D family	D_s family	J/ψ family	B, B_s and B_c family	Υ family
Value of c	1	1	0.931	0.931	1	0.994

In order to determine the parameters of the model, we fit 158 well-established meson states from the PDG [7] and some higher excited states [8]. Specifically, the mass spectra was used to determine the parameters a, a', b, b', $\bar{c}$, $\bar{c}'$, d, d', e, f, h and some of the mixing angles. However, some of the mixing angles need to be adjusted according to other experimental data, such as radiative decay widths that are discussed in the next section. Combining the mass spectra and the radiative decay widths, one can systematically determine all the parameters in the transitional theory to investigate whether the transitional theory with one more adjustable parameter c is systematically better than the $O(4)$ limit of the theory in fitting both mass spectra and radiative decay widths. The final values chosen for the parameters are give in table 2. A best fit requires that the new parameter c of the transitional theory take on different values according to the masses of the constituent quark and antiquark of the mesons. These results are shown in table 3. Quantum numbers J^{PC}, where P and C are the parity and charge conjugation quantum numbers, are assigned according to the PDG tables [7] and those shown in [8]. Other quantum numbers are determined for given J^{PC} according to masses and decay modes. Since the $O(4)$-$U(3)$ transition within the $U(4)$ model is the second-order quantum phase transition, changes in energies of meson masses and the corresponding wave functions in the $U(4)$ model with respect to variations of the parameters are quite smooth, which enables us to adjust the parameters in the model according to the corresponding experimental data with any fitting algorithm.

However, some of meson states are missing in the $U(4)$ model fit. These mesons are a) those with uncertain quantum number assignments in the D and D_s families like $D_1(2420)$, and those with non-$q\bar{q}$ content ones, and those involving substantial coupled-channel effects as for the $D\bar{D}$ and $B\bar{B}$ thresholds in the ψ and Υ families; b) $f_0(600)$ or σ, $f_0(980)$, $\pi_1(1400)$, $f_1(1420)$, $\pi_1(1600)$, $f_2(1430)$, $f_2(1565)$, $f_2(1640)$, $\eta(2225)$. Among these mesons, some of them are obviously not in pure $q\bar{q}$ configuration such as $f_0(600)$, $f_0(980)$, $\pi_1(1400)$, $\pi_1(1600)$, and some of them are not well established and need further confirmation [7,14–16]. Furthermore, as is shown in [7], decay modes of $\eta(1405)$ and $\eta(1475)$ are quite different. Only $\eta(1475)$ can be identified in the model calculations. In addition, though some meson states obviously contain non-$q\bar{q}$ contents or with multiquark and meson-molecule states [7,15,16], such as $a_0(980)$, $\eta(1295)$, $f_0(1370)$, $a_0(1450)$, $f_0(1500)$, $\pi(1800)$, $f_2(2340)$, $D_0^*(2308)$, $D_{sJ}^*(2317)$, and $D_{sJ}(2460)$, they can be fitted in the $U(4)$ model with certain deviations though some deviations are a little larger. As is known, the meson state can be written as

$$|\mathrm{M}\rangle = |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + \cdots + |q\bar{q}g\rangle + \cdots, \tag{15}$$

where $q\bar{q}$ denotes quark and antiquark, and g a gluon, etc. There are a lot of discussions on complicated non-$q\bar{q}$ contents in these mesons [16–20]. For example, in the scalar meson sector, for the mass spectrum below 2 GeV, the discussion in meson spectroscopy centers on the possibility of glueball, multiquark and dynamically generated meson-molecule states [17–19]. A similar situation may also occur in the D_s family. For example, the $D_s(2317)$ and the $D_s(2460)$ states are discussed as possible molecular states, since their masses lie considerably below the quark model predictions [16,20]. In the $U(4)$ model, however, only the quantized geometric excitations of the string-like $q\bar{q}$ can be described. Hence, the calculated results are solely based on the string-like $q\bar{q}$ configurations of these mesons, while other non-$q\bar{q}$ contents are not considered in the $U(4)$ model. Therefore, the fit shows that either some of these meson states may contain less non-$q\bar{q}$ contents, or those non-$q\bar{q}$ contents affect a little these meson masses. But we cannot draw a conclusion from the $U(4)$ model about them, which only contains string-like $q\bar{q}$ configurations. There are still some meson states predicted in the model calculations, but they have not been found in experiment up till now.

We use the mean relative deviation of masses with

$$\overline{\chi} = \frac{1}{\mathcal{N}} \sum_k \left| \frac{\left(M_k^{\mathrm{th}}\right) - \left(M_k^{\mathrm{exp}}\right)}{\left(M_k^{\mathrm{exp}}\right)} \right| \tag{16}$$

to measure the quality of fit, where $\mathcal{N}$ is the total number of mesons included in the fitting procedure.

Table 3 shows that the phase parameter c of the transitional theory can be chosen to have its $O(4)$ limit $c=1$ for mesons with light and strange quark content. Deviation from the $O(4)$ limit occurs when heavy quarks are involved. Since the $O(4)$ limit of the theory [5] is our starting point, we first fit other parameters for those mesons as shown in table 2, then adjust the parameter c to best fit the experimental data. Therefore, the parameters shown in table 2 with $c=1$ provide those used in the $O(4)$ limit theory proposed in [5]. It should be pointed out that the parameters used in the present fit shown in table 2 with $c=1$, namely in the $O(4)$ limit of the theory, can produce fits to the experimental data better than those used in [5]. However, our study shows that the parameter c deviating a little from 1 seems better in fitting the experimental values, especially for D_s, J/ψ, and Υ families. The exceptions are the D and B families, for which the confirmed experimental data are too few to determine the phase parameter c as shown in tables 7 and 9. Therefore, the c value for these two families were tentatively set to be 1 within the $O(4)$ limit. The results for the masses of 158 mesons fitted are shown in tables 4-13.

The results in tables 4 and 5 are divided into three parts. The first part lists π or η family members taken from the PDG [7]. The corresponding masses calculated by the quark potential model are taken from [9]. The second part of table 4 provides with masses of other relatively high excited meson states taken from [8]. Meson masses shown in the second parts of both table 4 and 5 were not considered in [5]. The third part list meson states that are either not in pure $q\bar{q}$ configuration or not found in experiment but predicted in the model calculations below the corresponding energy scale being fitted. Similarly, tables 6-13 are divided into two parts. The first part was also fit in [5], while the second part was not considered in [5]. For D_s, J/ψ, and Υ families, the $O(4)$ limit column is added with parameter c taken to be 1 in the theory. Higher excited states obtained in the quark potential model [9] are also included in our results. Such predicted states are labeled by quantum numbers $n^{2s+1}L_J$, where n is taken to be ξ in the transitional theory or v in the $O(4)$ limit of the theory with $c=1$.

It should be noted that the transitional theory is exactly the same as the $O(4)$ limit when $c=1$ with other parameters listed in table 2 unchanged. An additional $O(4)$ limit column is added in the corresponding tables only when the fitting procedure of the transitional theory yields a c value that is different from the $O(4)$ limit of the theory, namely, for the D_s, J/ψ, and Υ families, of which the corresponding value of the phase parameter c are given in table 3.

In tables 4-13, almost all $q\bar{q}$ meson masses from the PDG [7] and [8] are fit by the transitional theory. The results show that the theory reproduces the experimental data, especially those with pure $q\bar{q}$ contents, quite well. In order to show the advantage of the transitional theory, the mean relative deviations of masses of $q\bar{q}$ mesons calculated from the quark potential model [9], the $O(4)$ limit of the theory with phase parameter $c=1$ for all cases, and the transitional theory with different c values for different meson families are given in table 14. These results show that the transitional theory improves the predictions for the $q\bar{q}$ meson masses. To compare formulae which have a different number of parameters, the quality of the fits is also measured by the quantity

$$\bar{S}=\frac{1}{\mathcal{N}-n}\sum_k\left|\frac{\left(M_k^{\mathrm{th}}\right)-\left(M_k^{\mathrm{exp}}\right)}{\left(M_k^{\mathrm{exp}}\right)}\right|, \tag{17}$$

where n is the number of parameters in the formula. From the fitting results of mean relative deviations shown in table 14, it is clear that the transitional theory is better than the $O(4)$ limit of the theory and the quark potential model in describing the meson mass spectra. In tables 4-13, the entry with "–" indicates that the corresponding value is either not confirmed experimentally, or not calculated in the corresponding theory.

Most importantly, our study shows that the phase parameter c cannot be taken to be 1 when heavy quarks are involved in $q\bar{q}$ mesons. Deviations from the exactly linear "vibrational" Regge trajectories for the ψ and Υ meson families was noted in [5]. We now see that a possible explanation is $O(4)$ symmetry breaking. From a transitional theory point of view, the best-fit values of the phase parameter c for the ψ and Υ meson families are $c=0.994$ and $c=0.931$, respectively. In addition, one should take $c=0.931$ for the D_s family mesons. Therefore, the fitting procedure for the $q\bar{q}$ mesons of the transitional theory confirms the conclusion made in [5] that there is indeed $O(4)$ symmetry breaking when heavy quarks are involved in the $q\bar{q}$ mesons. Furthermore, our results show that the linear plus Coulomb $q\bar{q}$ potential $\frac{\alpha_s}{r}+\beta r$ suggested by QCD [21] cannot be expressed in terms of the p and s bosons in the $O(4)$ limit of the theory as accurately as when symmetry breaking is allowed. Although the $O(4)$ limit of the theory describes the relativistic light and strange meson spectra well, it deviates from the experimental data systematically in the non-relativistic regions. Our results show that the average relative deviation of meson masses is 2.88% for the light and K meson families for the transitional theory with $c=1$ giving the best results, while the average relative deviation of the meson masses for heavy mesons is 1.54% and 1.84%, respectively, in the transitional and $O(4)$ limit of the theory. This shows that $O(4)$ symmetry breaking should be taken into account for those non-relativistic $q\bar{q}$ mesons. As noted above, the phase parameter c was taken to be 1 for the D, B, B_s and B_c families due to the fact that there is too few experimental data to determine this parameter as is clearly shown in tables 7 and 11. More experimental data for these families is required to clarify this matter.

4 Radiative decays of $q\bar{q}$ mesons in the transitional theory

The analysis reported in the previous section shows that there are deviations from the $O(4)$ limit of the transitional theory when fitting the mass spectra of $q\bar{q}$ mesons in non-relativistic regions. In order to verify whether the theory

Table 4. Masses of $q\bar{q}$ mesons of the π family calculated with the transitional theory and in comparison with the experimental data for M (in GeV).

Meson	Experiment [7,8]	Quark model [9]	This theory	$\xi(v)$	L	S	J^{pc}
π	0.14 ± 0.00	0.15	0.140	0	0	0	0^{-+}
$\rho(770)$	0.78 ± 0.00	0.77	0.77	0	0	1	1^{--}
$b_1(1235)$	1.23 ± 0.00	1.22	1.22	0	1	0	1^{+-}
$a_1(1260)$	1.23 ± 0.04	1.24	1.28	0	1	1	1^{++}
$\pi(1300)$	1.30 ± 0.10	1.30	1.32	1	0	0	0^{-+}
$a_2(1320)$	1.32 ± 0.00	1.31	1.30	0	1	1	2^{++}
$\rho(1450)$	1.47 ± 0.03	1.45	1.39	1	0	1	1^{--}
$a_1(1640)$	1.64 ± 0.01	1.82	1.73	1	1	1	1^{++}
$\pi_2(1670)$	1.67 ± 0.02	1.68	1.60	0	2	0	2^{-+}
$\rho_3(1690)$	1.69 ± 0.01	1.68	1.67	0	2	1	3^{--}
$\rho(1700)$	1.70 ± 0.02	1.66	1.64	0	2	1	1^{--}
$a_2(1700)$	1.73 ± 0.03	1.82	1.74	1	1	1	2^{++}
$\rho(1900)$	~ 1.90	2.00	1.80	2	0	1	1^{--}
$\rho_3(1990)$	~ 1.99	2.13	2.03	1	2	1	3^{--}
$a_4(2040)$	2.01 ± 0.01	2.01	1.97	0	3	1	4^{++}
$\pi_2(2100)$	2.09 ± 0.03	2.13	1.98	1	2	0	2^{-+}
$\rho(2150)$	2.15 ± 0.02	–	2.13	3	0	1	1^{--}
$\rho_3(2250)$	~ 2.25	2.37	2.21	0	4	1	3^{--}
$\rho_5(2350)$	~ 2.35	2.30	2.23	0	4	1	5^{--}
$a_6(2450)$	~ 2.45	–	2.46	0	5	1	6^{++}
$a_1(1930)$	$1.93^{+0.03}_{-0.07}$	–	2.08	2	1	1	1^{++}
$\rho_2(1940)$	1.94 ± 0.04	2.15	2.02	1	2	1	2^{--}
$a_2(1950)$	$1.95^{+0.03}_{-0.07}$	2.05	1.95	0	3	1	2^{++}
$b_1(1960)$	1.96 ± 0.04	–	2.03	2	1	0	1^{+-}
$\rho(1970)$	1.97 ± 0.03	2.15	2.01	1	2	1	1^{--}
$a_2(1990)$	$1.99^{+0.02}_{-0.03}$	2.05	1.95	0	3	1	2^{++}
$a_0(2020)$	2.03 ± 0.03	–	2.07	2	1	1	0^{++}
$a_2(2030)$	2.03 ± 0.02	–	2.08	2	1	1	2^{++}
$a_3(2031)$	2.03 ± 0.01	2.05	1.96	0	3	1	3^{++}
$b_3(2032)$	2.03 ± 0.01	2.03	1.91	0	3	0	3^{+-}
$a_2(2175)$	2.18 ± 0.04	–	2.27	1	3	1	2^{++}
$\rho_2(2225)$	2.23 ± 0.04	–	2.32	2	2	1	2^{--}
$\rho_4(2240)$	2.23 ± 0.03	2.34	2.22	0	4	1	4^{--}
$b_1(2240)$	2.24 ± 0.04	–	2.33	3	1	0	1^{+-}
$b_3(2245)$	~ 2.25	–	2.24	1	3	0	3^{+-}
$a_2(2255)$	2.26 ± 0.02	–	2.37	3	1	1	2^{++}
$\rho(2265)$	2.27 ± 0.04	–	2.31	2	2	1	1^{--}
$a_1(2270)$	$2.27^{+0.06}_{-0.04}$	–	2.37	3	1	1	1^{++}
$a_3(2275)$	2.28 ± 0.04	–	2.27	1	3	1	3^{++}
$a_0(980)$	0.98 ± 0.00	1.09	1.27	0	1	1	0^{++}
$a_0(1450)$	1.47 ± 0.02	1.78	1.72	1	1	1	0^{++}
$\pi(1800)$	1.81 ± 0.01	1.88	1.75	2	0	0	0^{-+}
$1\ ^3D_2$	–	1.70	1.65	0	2	1	2^{--}
$4\ ^3P_0$	–	–	2.36	3	1	1	0^{++}

Table 5. Masses of $q\bar{q}$ mesons of the η family calculated with the transitional theory and in comparison with the experimental data for M (in GeV).

Meson	Experiment [7,8]	Quark model [9]	This theory	$\xi(v)$	L	S	J^{pc}
η	0.55 ± 0.00	0.49	0.55	0	0	0	0^{-+}
$\omega(782)$	0.78 ± 0.00	0.78	0.77	0	0	1	1^{--}
$\eta'(958)$	0.96 ± 0.00	0.93	0.96	0	0	0	0^{-+}
$\phi(1020)$	1.02 ± 0.00	1.02	1.06	0	0	1	1^{--}
$h_1(1170)$	1.17 ± 0.02	1.22	1.22	0	1	0	1^{+-}
$f_2(1270)$	1.28 ± 0.00	1.26	1.30	0	1	1	2^{++}
$f_1(1285)$	1.28 ± 0.00	1.24	1.28	0	1	1	1^{++}
$h_1(1380)$	1.39 ± 0.02	1.47	1.45	0	1	0	1^{+-}
$\omega(1420)$	1.42 ± 0.03	1.46	1.39	1	0	1	1^{--}
$\eta(1475)$	1.48 ± 0.00	1.63	1.56	1	0	0	0^{-+}
$f_1(1510)$	1.52 ± 0.01	1.48	1.52	0	1	1	1^{++}
$f_2'(1525)$	1.53 ± 0.01	1.53	1.54	0	1	1	2^{++}
$\eta_2(1645)$	1.62 ± 0.01	1.68	1.60	0	2	0	2^{-+}
$\omega(1650)$	1.65 ± 0.02	1.66	1.64	0	2	1	1^{--}
$\omega_3(1670)$	1.67 ± 0.00	1.68	1.67	0	2	1	3^{--}
$\phi(1680)$	1.68 ± 0.02	1.69	1.63	1	0	1	1^{--}
$f_0(1710)$	1.71 ± 0.01	1.78	1.72	1	1	1	0^{++}
$\eta(1760)$	1.76 ± 0.01	–	1.75	2	0	0	0^{-+}
$\phi_3(1850)$	1.85 ± 0.01	1.90	1.91	0	2	1	3^{--}
$\eta_2(1870)$	1.84 ± 0.01	1.89	1.84	0	2	0	2^{-+}
$f_2(1950)$	1.93 ± 0.01	2.05	1.95	0	3	1	2^{++}
$f_2(2010)$	$2.01^{+0.06}_{-0.08}$	–	2.07	2	1	1	2^{++}
$f_4(2050)$	2.03 ± 0.01	2.01	1.97	0	3	1	4^{++}
$f_0(2100)$	2.13 ± 0.00	–	2.08	2	1	1	0^{++}
$f_2(2150)$	2.16 ± 0.01	2.24	2.18	0	3	1	2^{++}
$f_4(2220)$	2.23 ± 0.00	2.20	2.22	0	3	1	4^{++}
$f_2(2300)$	2.30 ± 0.03	–	2.27	1	3	1	2^{++}
$f_4(2300)$	~ 2.30	–	2.28	1	3	1	4^{++}
$f_0(2330)$	~ 2.33	–	2.36	3	1	1	0^{++}
$f_6(2510)$	2.46 ± 0.05	–	2.46	0	5	1	6^{++}
$h_1(1595)$	1.59 ± 0.02	1.78	1.68	1	1	0	1^{+-}
$\omega_3(1945)$	1.95 ± 0.02	–	2.03	1	2	1	3^{--}
$\omega(1960)$	1.96 ± 0.03	–	2.01	1	2	1	1^{--}
$h_1(1965)$	1.97 ± 0.05	2.01	1.92	1	1	0	1^{+-}
$f_1(1971)$	1.97 ± 0.02	2.03	1.96	1	1	1	1^{++}
$f_2(1810)$	1.82 ± 0.01	1.82	1.74	1	1	1	2^{++}
$f_2(1910)$	1.92 ± 0.01	2.04	1.98	1	1	1	2^{++}
$\omega_2(1975)$	1.98 ± 0.02	–	2.02	1	2	1	2^{--}
$f_0(2020)$	1.99 ± 0.02	1.99	1.94	1	1	1	0^{++}
$h_3(2025)$	2.21 ± 0.02	2.22	2.16	0	3	0	3^{+-}
$f_3(2048)$	2.05 ± 0.01	2.05	1.96	0	3	1	3^{++}
$\omega_2(2195)$	2.20 ± 0.03	–	2.26	1	2	1	2^{--}
$\omega(2205)$	2.21 ± 0.03	–	2.13	3	0	1	1^{--}
$h_1(2215)$	2.22 ± 0.04	–	2.28	2	1	0	1^{+-}
$\omega_3(2255)$	2.26 ± 0.02	–	2.28	1	2	1	3^{--}
$h_3(2275)$	2.28 ± 0.03	–	2.24	1	3	0	3^{+-}
$\omega_3(2285)$	2.29 ± 0.06	–	2.33	2	2	1	3^{--}
$f_3(2303)$	2.30 ± 0.02	–	2.27	1	3	1	3^{++}
$f_1(2310)$	2.31 ± 0.06	–	2.32	2	1	1	1^{++}
$\eta(1295)$	1.29 ± 0.00	1.44	1.32	1	0	0	0^{-+}
$f_0(1370)$	1.35 ± 0.15	1.09	1.27	0	1	1	0^{++}
$f_0(1500)$	1.51 ± 0.01	1.36	1.49	0	1	1	0^{++}
$f_2(2340)$	2.34 ± 0.06	–	2.37	3	1	1	2^{++}
$s\bar{s}(3^3P_2)$	–	–	2.03	2	1	0	1^{+-}

Table 6. Masses of mesons of the K family calculated with the transitional theory and in comparison with the experimental data for M (in GeV).

Meson	Experiment [7]	Quark model [9]	This theory	$\xi(v)$	L	S	J^p
K	0.50 ± 0.00	0.47	0.48	0	0	0	0^-
$K^*(892)$	0.89 ± 0.00	0.90	0.91	0	0	1	1^-
$K_1(1270)$	1.27 ± 0.01	1.34	1.33	0	1	0	1^+
$K^*(1410)$	1.41 ± 0.02	1.58	1.51	1	0	1	1^-
$K_1(1400)$	1.40 ± 0.01	1.38	1.40	0	1	1	1^+
$K_0^*(1430)$	1.41 ± 0.01	1.24	1.37	0	1	1	0^+
$K_2^*(1430)$	1.43 ± 0.00	1.43	1.42	0	1	1	2^+
$K^*(1680)$	1.72 ± 0.03	1.78	1.75	0	2	1	1^-
$K_2(1770)$	1.77 ± 0.01	1.81	1.77	0	2	1	2^-
$K_3(1780)$	1.78 ± 0.01	1.79	1.79	0	2	1	3^-
$K_4^*(2045)$	2.05 ± 0.01	2.11	2.09	0	3	1	4^+
$K_0^*(1950)$	1.95 ± 0.01	1.89	1.83	1	1	1	0^+
$K_2^*(1980)$	1.97 ± 0.01	2.15	2.06	0	3	1	2^+
$K_5^*(2380)$	2.38 ± 0.01	2.39	2.36	0	4	1	5^-
$K(1460)$	~ 1.46	1.45	1.43	1	0	0	0^-
$K_2(1580)$	~ 1.58	1.78	1.72	0	2	0	2^-
$K_1(1650)$	1.65 ± 0.05	1.90	1.80	1	1	0	1^+
$K(1830)$	~ 1.83	2.02	1.86	2	0	0	0^-
$K_2(2250)$	2.25 ± 0.02	2.26	2.14	1	2	1	2^-
$K_3(2320)$	2.32 ± 0.02	–	2.36	1	3	0	3^+
$K_4(2500)$	2.49 ± 0.02	–	2.60	1	4	0	4^-
$2\,^3P_2$	–	1.94	1.86	1	1	1	2^+
$1\,^1F_3$	–	2.12	2.03	0	3	0	3^+
$1\,^3F_3$	–	2.15	2.08	0	3	1	3^+
$1\,^1G_4$	–	2.41	2.31	0	4	0	4^-
$1\,^3G_4$	–	2.44	2.34	0	4	1	4^-

Table 7. Masses of mesons of D family calculated by the transitional theory and in comparison to the experimental data M (in GeV).

Meson	Experiment [7]	Quark model [9]	This theory	$\xi(v)$	L	S	J^p
D	1.87 ± 0.00	1.88	1.89	0	0	0	0^-
D^*	2.01 ± 0.00	2.04	1.99	0	0	1	1^-
$D_2^*(2460)$	2.46 ± 0.00	2.50	2.41	0	1	1	2^+
$D_1(2420)$	2.43 ± 0.01	2.49	2.37	0	1	1	1^+
$D_0^*(2308)$	2.31 ± 0.02	2.44	2.32	0	1	0	1^+
$2\,^1S_0$	–	2.58	2.41	1	0	0	0^-
$2\,^3S_1$	–	2.64	2.50	1	0	1	1^-
$1\,^3D_1$	–	2.82	2.69	0	2	1	1^-
$1\,^3P_0$	–	2.40	2.32	0	1	1	0^+
$1\,^3D_3$	–	2.83	2.76	0	2	1	3^-
$1\,^3F_4$	–	3.11	2.97	0	3	1	1^+

also applies to the description of other physical quantities, in this section results for radiative decays of $q\bar{q}$ mesons in the transitional theory are considered.

We will adopt the definition of the transition operator for radiative decays in the $U(4)$ model proposed in [13],

$$T_\gamma = \frac{1}{\sqrt{k}} \times \sum_{i=1,2} \frac{e_i}{2m_i}\left[kg_i s_+^{(i)} + ikm_i\frac{\beta}{N}\nu^{(i)}\hat{D}_+\right]e^{-ik\beta\nu^{(i)}\hat{D}_0/N}, \quad (18)$$

where k is the momentum of the emitted photon, e_i, m_i are the charge and mass of quark i, g_i is the g factor, $s_+^i = s_x^i + is_y^i$ is the spin operator of the quark i, $\nu^i = (-)^{i+1}\frac{m_1m_2}{(m_1+m_2)m_i}$ $(i = 1, 2)$, and $\hat{D}_+ = -\sqrt{2}\hat{D}_1^{(1)}$ is the generator of $O(4)$ algebra with

$$\hat{D}_1^{(1)} = \left[p^+\tilde{s} + s^+\tilde{p}\right]_1^{(1)}, \quad (19)$$

β is a scale factor for coordinates of quark (antiquark) in mesons [17], and $\hat{D}_0 = [p^+\tilde{s} + s^+\tilde{p}]_0^{(1)}$.

Table 8. Masses of mesons of D_s family calculated by the transitional theory and in comparison to the experimental data M (in GeV).

Meson	Experiment [7]	Quark model [9]	$O(4)$ limit	This theory	$\xi(v)$	L	S	J^p
$D_s^{\pm}$	1.97 ± 0.00	1.98	2.04	1.97	0	0	0	0^-
D_s^*	2.11 ± 0.00	2.13	2.13	2.08	0	0	1	1^-
D_{s1}	2.54 ± 0.00	2.57	2.51	2.45	0	1	1	1^+
$D_{sJ}^*(2317)$	2.32 ± 0.00	2.48	2.50	2.40	0	1	1	0^+
$D_{sJ}(2460)$	2.46 ± 0.00	2.53	2.47	2.41	0	1	0	1^+
$2\ ^1S_0$	–	2.67	2.55	2.46	1	0	0	0^-
$2\ ^3S_1$	–	2.73	2.63	2.55	1	0	1	1^-
$1\ ^3D_1$	–	2.90	2.82	2.77	0	2	1	1^-
$1\ ^3P_2$	–	2.59	2.55	2.49	0	1	1	2^+
$1\ ^3D_2$	–	2.92	2.86	2.81	0	2	1	2^-
$1\ ^3F_4$	–	3.19	3.22	3.17	0	3	1	4^+

Table 9. Masses of mesons of B family calculated by the transitional theory and in comparison to the experimental data M (in GeV).

Meson	Experiment [7]	Quark model [9]	This theory	$\xi(v)$	L	S	J^p
B	5.28 ± 0.00	5.31	5.09	0	0	0	0^-
B^*	5.33 ± 0.00	5.37	5.18	0	0	1	1^-
$2\ ^1S_0$	–	5.90	5.52	1	0	0	0^-
$2\ ^3S_1$	–	5.93	5.60	1	0	1	1^-
$1\ ^3P_2$	–	5.80	5.53	0	1	1	2^+
$1\ ^3D_3$	–	6.11	5.85	0	2	1	3^-
$1\ ^3F_4$	–	6.36	6.16	0	3	1	4^+

Table 10. Masses of mesons of B_s family calculated by the transitional theory and in comparison to the experimental data M (in GeV).

Meson	Experiment [7]	Quark model [9]	This theory	$\xi(v)$	L	S	J^p
B_s^0	5.37 ± 0.00	5.39	5.24	0	0	0	0^-
B_s^*	5.42 ± 0.00	5.45	5.33	0	0	1	1^-
$2\ ^1S_0$	–	5.98	5.67	1	0	0	0^-
$2\ ^3S_1$	–	6.01	5.75	1	0	1	1^-
$1\ ^3P_2$	–	5.88	5.68	0	1	1	2^+
$1\ ^3D_3$	–	6.18	6.00	0	2	1	3^-
$1\ ^3F_4$	–	6.43	6.31	0	3	1	4^+

Table 11. Masses of mesons of B_c family calculated by the transitional theory and in comparison to the experimental data M (in GeV).

Meson	Experiment [7]	Quark model [9]	This theory	$\xi(v)$	L	S	J^p
$B_c^{\pm}$	6.40 ± 0.39	6.27	6.35	0	0	0	0^-
$1\ ^3S_1$	–	6.34	6.44	0	0	1	1^-
$2\ ^1S_0$	–	6.85	6.77	1	0	0	0^-
$2\ ^3S_1$	–	6.89	6.85	1	0	1	1^-
$1\ ^3P_2$	–	6.77	6.78	0	1	1	2^+
$1\ ^3D_3$	–	7.04	7.10	0	2	1	3^-
$1\ ^3F_4$	–	7.27	7.40	0	3	1	4^+

Table 12. Masses of mesons of the $c\bar{c}$ family calculated with the transitional theory and in comparison with the experimental data for M (in GeV).

Meson	Experiment [7]	Quark model [9]	$O(4)$ limit	This theory	$\xi(v)$	L	S	J^{pc}
$\eta_c(1s)$	2.98 ± 0.00	2.97	3.15	3.04	0	0	0	0^{-+}
$J/\psi(1s)$	3.10 ± 0.00	3.10	3.25	3.14	0	0	1	1^{--}
$\chi_{c0}(1P)$	3.42 ± 0.00	3.44	3.51	3.41	0	1	1	0^{++}
$\chi_{c1}(1P)$	3.51 ± 0.00	3.51	3.58	3.47	0	1	1	1^{++}
$\chi_{c2}(1P)$	3.56 ± 0.00	3.55	3.62	3.52	0	1	1	2^{++}
$\psi(2S)$	3.69 ± 0.00	3.68	3.70	3.58	1	0	1	1^{--}
$\psi(3770)$	3.77 ± 0.00	3.82	3.86	3.77	0	2	1	1^{--}
$\psi(4040)$	4.04 ± 0.01	4.10	4.10	3.96	2	0	1	1^{--}
$\psi(4160)$	4.16 ± 0.02	4.19	4.25	4.14	1	2	1	1^{--}
$\psi(4415)$	4.42 ± 0.01	4.45	4.45	4.30	3	0	1	1^{--}
$\eta_c(2S)$	3.65 ± 0.01	3.62	3.62	3.49	1	0	0	0^{-+}
$\psi(3836)$	3.84 ± 0.01	3.84	3.91	3.82	0	2	1	2^{--}
$X(3940)$	3.94 ± 0.01	3.96	3.96	3.84	1	1	0	1^{+-}
$Y(3940)$	3.94 ± 0.01	3.95	3.99	3.87	1	1	1	1^{++}
$X'_{c2}(3940)$	3.93 ± 0.00	3.98	4.03	3.92	1	1	1	2^{++}
$3\ ^1S_0$	–	4.06	4.02	3.88	2	0	0	0^{-+}
$1\ ^1P_1$	–	3.52	3.54	3.44	0	1	0	1^{+-}
$1\ ^1D_2$	–	3.84	3.88	3.79	0	2	0	2^{-+}
$1\ ^3D_3$	–	3.85	3.96	3.87	0	2	1	3^{--}
$1\ ^1F_3$	–	4.09	4.23	4.14	0	3	1	3^{++}
$1\ ^3F_4$	–	4.09	4.27	4.19	0	3	1	4^{++}

Table 13. Masses of mesons of the $b\bar{b}$ family calculated with the transitional theory and in comparison with the experimental data for M (in GeV).

Meson	Experiment [7]	Quark model [9]	$O(4)$ limit	This theory	$\xi(v)$	L	S	J^{pc}
$\Upsilon(1S)$	9.46 ± 0.00	9.46	9.64	9.61	0	0	1	1^{--}
$\chi_{b0}(1P)$	9.86 ± 0.00	9.85	9.83	9.81	0	1	1	0^{++}
$\chi_{b1}(1P)$	9.69 ± 0.00	9.88	9.90	9.87	0	1	1	1^{++}
$\chi_{b2}(1P)$	9.91 ± 0.00	9.90	9.96	9.93	0	1	1	2^{++}
$\Upsilon(2S)$	10.02 ± 0.00	10.00	10.03	10.00	1	0	1	1^{--}
$\chi_{b0}(2P)$	10.23 ± 0.00	10.23	10.22	10.19	1	1	1	0^{++}
$\chi_{b1}(2P)$	10.26 ± 0.00	10.25	10.28	10.25	1	1	1	1^{++}
$\chi_{b2}(2P)$	10.27 ± 0.00	10.26	10.34	10.31	1	1	1	2^{++}
$\Upsilon(3S)$	10.36 ± 0.00	10.35	10.41	10.38	2	0	1	1^{--}
$\Upsilon(4S)$	10.58 ± 0.00	10.63	10.76	10.73	3	0	1	1^{--}
$\Upsilon(10860)$	10.87 ± 0.01	10.88	11.09	11.06	4	0	1	1^{--}
$\Upsilon(11020)$	11.02 ± 0.01	11.10	11.41	11.38	5	0	1	1^{--}
$\eta_b(1S)$	9.30 ± 0.02	9.40	9.55	9.53	0	0	0	0^{-+}
$2\ ^1S_0$	–	9.98	9.95	9.92	1	0	0	0^{-+}
$1\ ^3D_1$	–	10.14	10.15	10.12	0	2	1	1^{--}
$1\ ^1D_2$	–	10.15	10.19	10.17	0	2	0	2^{-+}
$1\ ^3D_1$	–	10.15	10.21	10.18	0	2	1	2^{--}
$1\ ^3D_3$	–	10.16	10.27	10.25	0	2	1	3^{--}
$2\ ^1D_2$	–	10.45	10.51	10.48	1	2	0	2^{-+}
$1\ ^3F_2$	–	10.35	10.40	10.37	0	3	1	2^{++}
$1\ ^1F_3$	–	10.35	10.50	10.47	0	3	0	3^{+-}
$1\ ^3F_3$	–	10.35	10.52	10.49	0	3	1	3^{++}
$1\ ^3F_4$	–	10.36	10.57	10.55	0	3	1	4^{++}

Table 14. Mean relative deviations of the masses of $q\bar{q}$ mesons. The first row, $\overline{\chi}$, gives the mean relative deviation of the masses of mesons calculated according to (16), and the second row, $\overline{S}$, is calculated according to (17).

Model	Quark model [9]	$O(4)$ limit	This theory
$\overline{\chi}$	3.05%	2.66%	2.58%
$\overline{S}$	3.57%	3.04%	2.97%

Table 15. Expressions for radiative decay widths of the light mesons.

Selection rules $\Delta S = 1$, $\Delta L = 0$

$$\Gamma_{\xi=0}(\rho^{\pm} \longrightarrow \gamma\pi^{\pm}) = \frac{\mu_{ud}^2 k^3}{27\pi}\left|\sum_v (C_v^0)^2\, F_0(kv)\right|^2$$

$$\Gamma_{\xi=0}(\rho^{0} \longrightarrow \gamma\eta) = \frac{\mu_{ud}^2 k^3}{3\pi}\left|\sum_v (C_v^0)^2\, F_0(kv) y_{11}^P\right|^2$$

$$\Gamma_{\xi=0}(\omega \longrightarrow \gamma\pi^{0}) = \frac{\mu_{ud}^2 k^3}{3\pi}\left|\sum_v (C_v^0)^2\, F_0(kv) y_{11}^V\right|^2$$

$$\Gamma_{\xi=0}(\omega \longrightarrow \gamma\eta) = \frac{\mu_{ud}^2 k^3}{27\pi}\left|\sum_v (C_v^0)^2\, F_0(kv)\left[y_{11}^P y_{11}^V - 2\frac{m_{ud}}{m_s} y_{12}^P y_{12}^V + 4\frac{m_{ud}}{m_c} y_{13}^P y_{13}^V\right]\right|^2$$

$$\Gamma_{\xi=0}(\eta' \longrightarrow \gamma\rho^{0}) = \frac{\mu_{ud}^2 k^3}{\pi}\left|\sum_v (C_v^0)^2\, F_0(kv) y_{21}^P\right|^2$$

$$\Gamma_{\xi=0}(\eta' \longrightarrow \gamma\omega) = \frac{\mu_{ud}^2 k^3}{9\pi}\left|\sum_v (C_v^0)^2\, F_0(kv)\left[y_{21}^P y_{11}^V - 2\frac{m_{ud}}{m_s} y_{22}^P y_{12}^V + 4\frac{m_{ud}}{m_c} y_{23}^P y_{13}^V\right]\right|^2$$

$$\Gamma_{\xi=0}(\phi \longrightarrow \gamma\pi^{0}) = \frac{\mu_{ud}^2 k^3}{3\pi}\left|\sum_v (C_v^0)^2\, F_0(kv) y_{21}^V\right|^2$$

$$\Gamma_{\xi=0}(\phi \longrightarrow \gamma\eta) = \frac{\mu_{ud}^2 k^3}{27\pi}\left|\sum_v (C_v^0)^2\, F_0(kv)\left[y_{11}^P y_{21}^V - 2\frac{m_{ud}}{m_s} y_{12}^P y_{22}^V + 4\frac{m_{ud}}{m_c} y_{13}^P y_{23}^V\right]\right|^2$$

$$\Gamma_{\xi=0}(\phi \longrightarrow \gamma\eta') = \frac{\mu_{ud}^2 k^3}{27\pi}\left|\sum_v (C_v^0)^2\, F_0(kv)\left[y_{21}^P y_{21}^V - 2\frac{m_{ud}}{m_s} y_{22}^P y_{22}^V + 4\frac{m_{ud}}{m_c} y_{23}^P y_{23}^V\right]\right|^2$$

Selection rules $\Delta S = 1$, $\Delta L = 1$

$$\Gamma_{\xi=0}({a_1}^{\pm} \longrightarrow \gamma\pi^{\pm}) = \frac{\mu_{ud}^2 k^3}{2\pi}\left|\sum_v (C_v^0)^2\, F_1(kv)\right|^2$$

$$\Gamma_{\xi=0}({a_2}^{\pm} \longrightarrow \gamma\pi^{\pm}) = \frac{3\mu_{ud}^2 k^3}{10\pi}\left|\sum_v (C_v^0)^2\, F_1(kv)\right|^2$$

Selection rules $\Delta S = 0$, $\Delta L = 1$

$$\Gamma_{\xi=0}({b_1}^{\pm} \longrightarrow \gamma\pi^{\pm}) = \frac{e^2 k^3}{27\pi}\left|\sum_v (C_v^0)^2\, G_1(kv)\right|^2$$

Since the radiative decay width in [13] was defined in the $O(4)$ limit of the theory, the meson state vectors $|N,\xi,L,S,J,M_J\rangle$ shown in (5) will be expanded in terms of those in the $O(4)$ limit of the theory with

$$\big|N,\xi,L,S,J,M_J\big\rangle = \sum_v C_v^{\xi}\big|N,v,L,S,J,M_J\big\rangle, \qquad (20)$$

where v is the quantum number that labels the irreducible representations of $O(4)$ and C_v^{ξ} is the expansion coefficient. The selection rule of the transition operator (18) for the quantum number v in the $O(4)$ limit is $\Delta v = 0$. It follows that the radiative decay width in the transitional theory can be expressed as

$$\Gamma\big(M_\xi \longrightarrow M'_{\xi'} + \gamma\big) = \frac{2k^2}{(2J+1)\pi}\sum_{M_J,M_J}\left|\sum_v C_v^{\xi} C_v^{\xi'}\,\langle M'_v|\hat{T}_\gamma|M_v\rangle\right|^2, \qquad (21)$$

where M_v and ${M_v}'$ stand for a set of quantum numbers (N,v,L,S,J,M_J) and (N,v,L',S',J',M'_J) in the $O(4)$ limit, respectively, and the matrix elements $\langle {M_v}'|\hat{T}_\gamma|M_v\rangle$ in the $O(4)$ limit of the theory used in (21) are given in [13]. Expressions for radiative decay widths for light, strange, and heavy mesons with selection rules for orbital angular momentum and spin are given in tables 15-17.

Table 16. Expressions for radiative decay widths of the J/ψ mesons.

Selection rules $\Delta S = 1,\ \Delta L = 0$
$\Gamma_{\xi=0}(J/\psi \longrightarrow \gamma\eta_c) = \frac{\mu_{ud}^2 k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, F_0(kv)\left[y_{31}^P y_{31}^V - 2\frac{m_{ud}}{m_s} y_{32}^P y_{32}^V + 4\frac{m_{ud}}{m_c} y_{33}^P y_{33}^V\right]\right\vert^2$
OZI forbidden decays with selection rules $\Delta S = 1,\ \Delta L = 0$
$\Gamma_{\xi=0}(J/\psi \longrightarrow \gamma\eta') = \frac{\mu_{ud}^2 k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, F_0(kv)\left[y_{21}^P y_{31}^V - \frac{2m_u}{m_s} y_{22}^P y_{32}^V + \frac{4m_u}{m_c} y_{23}^P y_{23}^V\right]\right\vert^2$
$\Gamma_{\xi=0}(J/\psi \longrightarrow \gamma\eta) = \frac{\mu_{ud}^2 k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, F_0(kv)\left[y_{11}^P y_{31}^V - \frac{2m_u}{m_s} y_{12}^P y_{32}^V + \frac{4m_u}{m_c} y_{23}^P y_{13}^V\right]\right\vert^2$
$\Gamma_{\xi=0}(J/\psi \longrightarrow \gamma\pi^0) = \frac{\mu_{ud}^2 k^3}{3\pi}\left\vert \sum_v (C_v^0)^2\, F_0(kv)(y_{31}^V)\right\vert^2$
Selection rules $\Delta S = 0,\ \Delta L = 1$
$\Gamma_{\xi=0}(\chi_{c0} \longrightarrow \gamma J/\psi) = \frac{16e^2k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, G_1(kv) y_{33}^V\right\vert^2$
$\Gamma_{\xi=0}(\chi_{c1} \longrightarrow \gamma J/\psi) = \frac{16e^2k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, G_1(kv) y_{33}^V\right\vert^2$
$\Gamma_{\xi=0}(\chi_{c2} \longrightarrow \gamma J/\psi) = \frac{16e^2k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\, G_1(kv) y_{33}^V\right\vert^2$

Table 17. Expressions for radiative decay widths of the D_s family mesons.

Selection rules $\Delta S = 1,\ \Delta L = 0$
$\Gamma_{\xi=0}(D_s^{*+} \longrightarrow \gamma D_S^+) = \frac{\mu_c^2 k^3}{27\pi}\left\vert \sum_v (C_v^0)^2\ \left[2F_0\left[\frac{2m_s kv}{m_s+m_c}\right] - \frac{m_c}{m_s} F_0\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$
Selection rules $\Delta S = 0,\ \Delta L = 1$
$\Gamma_{\xi=0}(D_{SJ}^*(2317)^+ \longrightarrow \gamma D_s^{*+}) = \frac{k^3e^2}{36\pi}\left\vert \sum_v (C_v^0)^2\ \left[2G_1\left[\frac{2m_s kv}{m_s+m_c}\right] - G_1\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$
$\Gamma_{\xi=0}(D_{s1}(2536)^+ \longrightarrow \gamma D_s^{*+}) = \frac{2k^3e^2}{27\pi}\left\vert \sum_v (C_v^0)^2\ \left[2G_1\left[\frac{2m_s kv}{m_s+m_c}\right] - G_1\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$
$\Gamma_{\xi=0}(D_{SJ}(2460)^+ \longrightarrow \gamma D_s^{+}) = \frac{k^3e^2}{27\pi}\left\vert \sum_v (C_v^0)^2\ \left[2G_1\left[\frac{2m_s kv}{m_s+m_c}\right] - G_1\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$
Selection rules $\Delta S = 1,\ \Delta L = 1$
$\Gamma_{\xi=0}(D_{SJ}^*(2317)^+ \longrightarrow \gamma D_s^+) = 0$
$\Gamma_{\xi=0}(D_{s1}(2536)^+ \longrightarrow \gamma D_s^{+}) = \frac{\mu_c^2 k^3}{18\pi}\left\vert \sum_v (C_v^0)^2\ \left[2F_1\left[\frac{2m_s kv}{m_s+m_c}\right] + \frac{m_c}{m_s} F_1\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$
$\Gamma_{\xi=0}(D_{SJ}(2460)^+ \longrightarrow \gamma D_S^{*+}) = \frac{\mu_c^2 k^3}{9\pi}\left\vert \sum_v (C_v^0)^2\ \left[2F_1\left[\frac{2m_s kv}{m_s+m_c}\right] + \frac{m_c}{m_s} F_1\left[\frac{2m_c kv}{m_s+m_c}\right]\right]\right\vert^2$

In tables 15-17, $\mu_{ud} = \frac{ge}{2m_{ud}}$, $\mu_c = \frac{ge}{2m_c}$, $F_0(kv) = j_0(\frac{\beta kv(N-2v)}{N})$ is the spherical Bessel function, $F_1(kv) = j_1(\frac{\beta kv(N-2v)}{N})$, and $G_1(kv) = \frac{\beta v}{\sqrt{3}}[j_0(\frac{\beta kv(N-2v)}{N}) + j_2(\frac{\beta kv(N-2v)}{N})]$ with the following functional forms:

$$j_0(x) = \frac{\sin x}{x}\,,$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}\,, \tag{22}$$
$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x.$$

By using the results listed in tables 15-17, radiative decay widths for mesons were fit and compared with the experimental data taken from [7,13] except those in the D_s family. The results are shown in table 18. Since most of the parameters in the model were already determined in the corresponding mass spectra analysis, the adjustable parameters are only the effective g-factor and the scale factor β for coordinates of the quark (antiquark) in mesons. Similar to the fit for the masses of mesons, we used the mean relative deviation

$$\overline{\chi'} = \frac{1}{\mathcal{N}}\sum_k \left\vert \frac{\Gamma_k^{\rm th} - \Gamma_k^{\rm exp}}{\Gamma_k^{\rm exp}}\right\vert \tag{23}$$

Table 18. Radiative decay widths of mesons (in keV). The first part of the table gives decay widths for light mesons and the second part gives results for selected heavy mesons.

Decay mode	Experiment [7,13]	This theory	Quark model [9]	$O(4)$ limit
$\rho^{\pm} \longrightarrow \gamma\pi^{\pm}$	67.5 ± 7.5	73.831	68	
$\rho^{0} \longrightarrow \gamma\eta$	46.5 ± 17	64.3	44	
$\omega \longrightarrow \gamma\pi^{0}$	767 ± 51	702	646	
$\omega \longrightarrow \gamma\eta$	4.3 ± 1.9	7.3	5.4	
$\eta' \longrightarrow \gamma\rho^{0}$	59 ± 9	61	137	
$\eta' \longrightarrow \gamma\omega$	6.1 ± 1.2	7.2	13	
$\phi \longrightarrow \gamma\pi^{0}$	5.28 ± 0.67	8.64	1.3	
$\phi \longrightarrow \gamma\eta$	55.4 ± 3.5	37.8	66	
$\phi \longrightarrow \gamma\eta'$	0.26 ± 0.06	0.34	0.3	
${a_1}^{\pm} \longrightarrow \gamma\pi^{\pm}$	640 ± 246	308	314	
${a_2}^{\pm} \longrightarrow \gamma\pi^{\pm}$	280 ± 33	231	303	
${b_1}^{\pm} \longrightarrow \gamma\pi^{\pm}$	227.2 ± 56.8	114.8	397	
$K^{*0} \longrightarrow \gamma K^{0}$	116.8 ± 10	115.4	95	
$K^{*\pm} \longrightarrow \gamma K^{\pm}$	50 ± 5	79	66	
${K_2}^{*\pm} \longrightarrow \gamma K^{\pm}$	236.4 ± 49.3	176.2	230	
$J/\psi \longrightarrow \gamma\eta_c$	1.131 ± 0.348	1.036	1.9	1.036
$J/\psi \longrightarrow \gamma\eta'$	0.37497 ± 0.0261	0.30063	0.068	0.29968
$J/\psi \longrightarrow \gamma\eta$	0.07482 ± 0.00696	0.07511	0.009	0.07475
$J/\psi \longrightarrow \gamma\pi^{0}$	0.003393 ± 0.001131	0.003139	0.005	0.003131
$\chi_{c0} \longrightarrow \gamma J/\psi$	120.36 ± 27.54	145.76		145.99
$\chi_{c1} \longrightarrow \gamma J/\psi$	287.56 ± 29.44	305.20		305.68
$\chi_{c2} \longrightarrow \gamma J/\psi$	426.72 ± 41.6	409.66		410.30

Table 19. Radiative decay widths of D_s family mesons (in keV). Our results are compared with those from experimental limits reported by CLEO [22] and Belle [23], quark models with a mixture of conventional P-wave quark-antiquark states and four-quark components [16], and two different quark models with only $q\bar{q}$ components [20,24]. Theoretical predictions from light-cone QCD sum rules [25] and vector meson dominance [26] are also quoted.

			Quark models			Experiments		Other approaches	
Decay mode	This theory	$O(4)$ limit	Ref. [16]	Ref. [24]	Ref. [20]	CLEO [22]	Belle [23]	Ref. [25]	Ref. [26]
$D_s^{*+} \longrightarrow \gamma D_s^{+}$	0.3501	0.350235	–	–	0.125–0.19	–	–	–	–
$D_{sJ}^{*}(2317)^{+} \longrightarrow \gamma D_s^{*+}$	0.896946	0.898339	1.6	1.7	1.9	< 0.59	< 1.8	0.4–0.6	0.85
$D_{sJ}^{*}(2317)^{+} \longrightarrow \gamma D_s^{+}$	0.0	0.0	0.0	0.0	0.0	< 0.52	< 0.5	0.0	0.0
$D_{sJ}(2460)^{+} \longrightarrow \gamma D_s^{*+}$	2.1475	2.15388	0.06	4.7	5.5	< 1.6	< 3.1	0.6–1.1	1.5
$D_{sJ}(2460)^{+} \longrightarrow \gamma D_s^{+}$	6.8418	6.85	6.7	5.1	6.2	< 4.9	$5.5 \pm 1.3 \pm 0.8$	19–29	3.3
$D_{s1}(2536)^{+} \longrightarrow \gamma D_s^{+}$	8.92214	8.94787	–	–	9.0	–	–	–	–
$D_{s1}(2536)^{+} \longrightarrow \gamma D_s^{*+}$	11.4388	11.4544	–	–	9.2	–	–	–	–

to measure the quality of the fit to the radiative decay widths of the mesons. In the calculation the effective g-factor was taken to be $g = 0.6$, while β was taken to be $\beta = \beta_1 = 0.50\,\mathrm{fm}$ for light and strange mesons and $\beta = \beta_2 = 0.38\,\mathrm{fm}$ for heavy mesons. Accordingly, we get $\overline{\chi'} = 24.49\%$ in the transitional theory, 24.51% in the $O(4)$ limit of the theory. Furthermore, the results of our calculation show that the mean relative deviation of the radiative decay widths for light and strange mesons in the transitional theory is 31.43%, which is exactly the same as for the $O(4)$ limit of the theory. For heavy mesons, it is 9.62% in the transitional theory, and 9.68% in the $O(4)$ limit. Hence, the transitional theory seems to yield slightly better results than the $O(4)$ limit of the theory when fitting these radiative decay widths. However, the mean relative deviation in the quark potential model taken from [10] is 39.0% for light and strange mesons and 71.3% for heavy mesons. Therefore, the results indicate that the $U(4)$ model is better than the quark potential model when both mass spectra and the radiative decay widths are taken into account. While the results also show that the transitional theory yields an overall better fit to heavy-meson data than the $O(4)$ limit of the theory, especially when both mass spectra and radiative decay widths are taken into account, there is insufficient experimental data for some families of heavy mesons to conclude that this holds in all cases.

Due to large uncertainty of the experimental results, we compare in table 19 our results for the radiative transitions of the $D^*_{sJ}(2317)$ and $Ds_{sJ}(2460)$ with different theoretical approaches and the experimental limits reported by CLEO [22] and Belle [23]. Our results are similar to those obtained from quark models with only $q\bar{q}$ components [20, 24], while different from the quark model with a mixture of conventional P-wave quark-antiquark states and four-quark components [16], especially for the decay $D_{sJ}(2460)^+ \longrightarrow \gamma D^{*+}_s$. In addition, there is only a slight difference between the results from the transitional theory and those from the $O(4)$ limit as shown in the table 19. Theoretical predictions from light-cone QCD sum rules [25] and vector meson dominance [26] are also quoted.

5 Conclusion

In this paper, the general transitional description of the mass spectra and radiative decay widths for $q\bar{q}$ mesons in the $U(4)$ model according to the spectrum generating algebra $U(4) \otimes SU_S(2) \otimes SU_f(6) \otimes SU_c(3)$ is considered. Mass spectra and radiative decay widths are calculated and compared with the corresponding available experimental data. The results show that the $O(4)$ limit of the theory describes both light and strange $q\bar{q}$ mesons rather well. The only possible deviations from the $O(4)$ limit may occur in the description of heavy mesons. Fitting both the mass spectra and the radiative decay widths in the $U(4)$ model shows that deviations from $O(4)$ limit are possible in non-relativistic regions, which confirms the early observation made in [5] that the $O(4)$ symmetry breaking may occur when heavy quarks are involved in these $q\bar{q}$ mesons. Since the transitional theory only affects the spatial excitations, it is reasonable that there is only a little improvement in fitting the radiative decay widths. However, since there are too few experimental data to determine the phase parameter c in the transitional theory for the D and B meson families, more experimental data is needed in order to draw a general conclusion. Though there are only a few experimental values for radiative decay widths available, the results that are known do show that the transitional theory describes heavy mesons better than the $O(4)$ limit, both for the mass spectra and radiative decay widths. Nonetheless, the results confirm that the $O(4)$ limit of the theory yields a good description of light and strange mesons, one that cannot be improved upon by moving away from the $c = 1$ limit of the transitional theory. More generally, this study shows that both the quark model [9, 20, 24] and the $O(4)$ limit can be regarded as reasonable simple models for a description of most $q\bar{q}$ mesons, light, strange and heavy.

In summary, our study on the transitional description of the mass spectra and radiative decay widths for $q\bar{q}$ mesons in the $U(4)$ model with up to date experimental data confirms that the $O(4)$ dynamical symmetry is the most important ingredient in the $U(4)$ model, which agrees with the early observations [5, 13] made by Iachello *et al.* The possible deviations from the $O(4)$ limit may occur only when heavy quarks are involved in these $q\bar{q}$ mesons, which is also a suggestion made in the early work [5]. Furthermore, the results reported in this paper, together with those shown in [5] and [13], indicates that the $U(4)$ model is the simplest and reasonable algebraic model in describing spacial excitations of the string-like $q\bar{q}$ mesons. However, in order to describe non-$q\bar{q}$ contents in mesons, other configurations, such as glueball, multi-quark, and meson-molecule states should also be considered in the model as an extension.

The transitional theory can also be used to study strong decays of $q\bar{q}$ mesons as has been done for the $O(4)$ limit of the theory [27], and more complicated string-like excitations in hadron systems, such as baryons [28, 29], and the illusive pentaquarks [30]. Work in this direction is in progress.

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